

# Casimir forces between arbitrary compact objects

T. Emig,<sup>1</sup> N. Graham,<sup>2,3</sup> R. L. Jaffe,<sup>3</sup> and M. Kardar<sup>4</sup>

<sup>1</sup>*Laboratoire de Physique Théorique et Modèles Statistiques,  
CNRS UMR 8626, Université Paris-Sud, 91405 Orsay, France*

<sup>2</sup>*Department of Physics, Middlebury College, Middlebury, VT 05753*

<sup>3</sup>*Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics,  
Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

<sup>4</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*  
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We develop an exact method for computing the Casimir energy between arbitrary compact objects, either dielectrics or perfect conductors. The energy is obtained as an interaction between multipoles, generated by quantum current fluctuations. The objects' shape and composition enter only through their scattering matrices. The result is exact when all multipoles are included, and converges rapidly. A low frequency expansion yields the energy as a series in the ratio of the objects' size to their separation. As an example, we obtain this series for two dielectric spheres and the full interaction at all separations for perfectly conducting spheres.

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The electromagnetic (EM) force between neutral bodies is governed by the coordinated dance of fluctuating charges [1]. At the atomic scale, this attractive interaction appears in the guises of van der Waals, Keesom, Debye, and London forces. The collective behavior of condensed atoms is better formulated in terms of dielectric properties. In 1948, Casimir computed the force between two parallel metallic plates by focusing on the quantum fluctuations of the EM field between the two plates [2]. This was extended by Lifshitz to dielectric plates, accounting for the fluctuating fields in the media [3]. The force between atoms at asymptotically large distances was computed by Casimir and Polder [4] and related to the atoms' polarizabilities. For compact objects, such as two spheres, Feinberg and Sucher [5] generalized this work to include magnetic effects.

In this Letter we obtain the EM Casimir interaction between compact objects *at arbitrary separations* [6], and determine explicitly the dependence on shape and material properties [7]. In a qualitative sense, our approach is similar to a multipole expansion for the fluctuating sources. The dependence on shape and material appears through the susceptibility to current fluctuations, and is related to the scattering of EM waves by the object. While the scattering matrix is in principle complicated, there are tools for computing it and it is known for certain geometries. As an example, we compute the EM force between two dielectric spheres at any separation.

Earlier studies of the Casimir force between compact objects include a multiple reflection formalism [8], which in principle could be applied to perfect conductors of arbitrary shape. A formulation of the Casimir energy of compact objects in terms of their scattering matrices, for a scalar field coupled to a dielectric background, is introduced in Ref. [9], where it is suggested that it can also be extended to the EM case.

Many of our results can be derived by either Green's function or path integral methods. We shall sketch the latter derivation — due to the letter format only the key steps are outlined, and details are left for a more complete exposition [10]. Note first that since the objects are fixed in time, the action is diagonal in the frequency  $k$ . Therefore in all subsequent steps we can treat each frequency independently, and integrate over  $k$  at the end. The Casimir energy can be associated with modifications of gauge field fluctuations due to constraints imposed by boundary conditions at the material objects. An alternative and equivalent description, stressed by Schwinger [11], is to attribute the Casimir interaction to fluctuating current and charge densities  $\mathbf{J}$ ,  $\varrho$  inside the objects. In the latter formulation, the EM gauge and scalar potential  $[A(\mathbf{x}, t), \Phi(\mathbf{x}, t)]$  are given for each source configuration by the classical solutions, which in Lorentz gauge read

$$[A(\mathbf{x}), \Phi(\mathbf{x})] = \int d\mathbf{x}' G_0(\mathbf{x}, \mathbf{x}') [\mathbf{J}(\mathbf{x}'), \varrho(\mathbf{x}')] , \quad (1)$$

with  $G_0(\mathbf{x}, \mathbf{x}') = e^{ik|\mathbf{x}-\mathbf{x}'|}/(4\pi|\mathbf{x}-\mathbf{x}'|)$ . For path integral quantization, we integrate over all allowed configurations of the fluctuating currents, weighted by the appropriate action. The Lagrangian for a collection of currents in vacuum is the kinetic energy  $\frac{1}{2}\mathbf{J}\mathbf{A}$  minus the potential energy  $\frac{1}{2}\varrho\Phi$ . This yields, using Eq. (1) and the continuity equation  $\nabla\mathbf{J} = ik\varrho$ , the action  $S[\mathbf{J}] = \int (dk/4\pi)(S_k[\mathbf{J}] + S_k^*[\mathbf{J}])$  for the current densities  $\{\mathbf{J}_\alpha\}$  on the objects with

$$S_k[\{\mathbf{J}_\alpha\}] = \frac{1}{2} \int d\mathbf{x} d\mathbf{x}' \sum_{\alpha\beta} \mathbf{J}_\alpha^*(\mathbf{x}) \mathcal{G}_0(\mathbf{x}, \mathbf{x}') \mathbf{J}_\beta(\mathbf{x}') , \quad (2)$$

where  $\mathcal{G}_0(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') - \frac{1}{k^2} \nabla \otimes \nabla' G_0(\mathbf{x}, \mathbf{x}')$  is the tensor Green function. Next we must constrain the currents to be *induced* sources that depend on shape and material of the objects. Formally this is achieved by integrating over currents, inserting constraints to ensure

that the currents in vacuum simulate the correct induction of microscopic polarization  $\mathbf{P}_\alpha$  and magnetization  $\mathbf{M}_\alpha$  (from all multipoles) inside the dielectric objects in response to an incident wave.

Let us consider one object. First, the induced current is  $\mathbf{J}_\alpha = -ik\mathbf{P}_\alpha + \nabla \times \mathbf{M}_\alpha$ , and since  $\mathbf{P}_\alpha = (\epsilon_\alpha - 1)\mathbf{E}$ ,  $\mathbf{M}_\alpha = (1 - 1/\mu_\alpha)\mathbf{B}$ , it can be expressed in terms of the total fields  $\mathbf{E}$ ,  $\mathbf{B}$  inside the object as

$$\mathbf{J}_\alpha = -ik(\epsilon_\alpha - 1)\mathbf{E} + \nabla \times [(1 - 1/\mu_\alpha)\mathbf{B}]. \quad (3)$$

Second, the total field inside the object must consist of the field generated by  $\mathbf{J}_\alpha$  and the incident field  $\mathbf{E}_0(\{\mathbf{J}_\alpha, \mathbb{S}^\alpha\}, \mathbf{x})$  that has to impinge on the object to induce  $\mathbf{J}_\alpha$ , so that

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}_0(\{\mathbf{J}_\alpha, \mathbb{S}^\alpha\}, \mathbf{x}) + ik \int d\mathbf{x}' \mathcal{G}_0(\mathbf{x}, \mathbf{x}') \mathbf{J}_\alpha(\mathbf{x}'). \quad (4)$$

The incident field depends on the current density to be induced and on the scattering matrix  $\mathbb{S}^\alpha$  of the object, which connects the incident wave to the scattered wave. It is fully specified by the multipole moments of  $\mathbf{J}_\alpha$  (see below for details). Substituting Eq. (4) and  $\mathbf{B} = (1/ik)\nabla \times \mathbf{E}$  into Eq. (3) yields a self-consistency condition that constrains the current  $\mathbf{J}_\alpha$ . If one writes this condition as  $\mathcal{C}_\alpha[\mathbf{J}_\alpha] = 0$  for each object, the functional integration over the currents constrained this way for all objects yields the partition function

$$\mathcal{Z} = \int \prod_\alpha \mathcal{D}\mathbf{J}_\alpha \prod_{\mathbf{x} \in V_\alpha} \delta(\mathcal{C}_\alpha[\mathbf{J}_\alpha(\mathbf{x})]) \exp(iS[\{\mathbf{J}_\alpha\}]). \quad (5)$$

It is instructive to look at two compact objects at a distance  $L$ , measured between the (arbitrary) origins  $\mathcal{O}_\alpha$  inside the objects. In this case the action of Eq. (2) is

$$\begin{aligned} S_k[\{\mathbf{J}_\alpha\}] &= \frac{1}{2} \sum_{\alpha \neq \beta} \int d\mathbf{x}_\alpha \mathbf{J}_\alpha^*(\mathbf{x}_\alpha) \frac{1}{ik} \mathbf{E}_\beta(\mathbf{x}_\alpha - L_\alpha \hat{\mathbf{z}}) \\ &+ \frac{1}{2} \sum_\alpha \int d\mathbf{x}_\alpha d\mathbf{x}'_\alpha \mathbf{J}_\alpha^*(\mathbf{x}_\alpha) \mathcal{G}_0(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) \mathbf{J}_\alpha(\mathbf{x}'_\alpha), \end{aligned} \quad (6)$$

where we have substituted the electric field  $\mathbf{E}_\alpha(\mathbf{x}_\alpha) = ik \int d\mathbf{x}'_\alpha \mathcal{G}_0(\mathbf{x}_\alpha, \mathbf{x}'_\alpha) \mathbf{J}_\alpha(\mathbf{x}'_\alpha)$  and the fields are measured now in local coordinates so that  $\mathbf{x} = \mathcal{O}_\alpha + \mathbf{x}_\alpha$ , and  $L_\alpha = L$  ( $-L$ ) for  $\alpha = 1(2)$ . The off-diagonal terms in Eq. (6) represent the interaction between the currents on the two materials. A natural way to decompose the interaction between charges is to use the multipole expansion. For each body we define magnetic and electric multipoles as

$$\begin{aligned} Q_{M,lm}^\alpha &= \frac{k}{\lambda} \int d\mathbf{x}_\alpha \mathbf{J}_\alpha(\mathbf{x}_\alpha) \nabla \times [\mathbf{x}_\alpha j_l(kr_\alpha) Y_{lm}^*(\hat{\mathbf{x}}_\alpha)] \\ Q_{E,lm}^\alpha &= \frac{1}{\lambda} \int d\mathbf{x}_\alpha \mathbf{J}_\alpha(\mathbf{x}_\alpha) \nabla \times \nabla \times [\mathbf{x}_\alpha j_l(kr_\alpha) Y_{lm}^*(\hat{\mathbf{x}}_\alpha)], \end{aligned} \quad (7)$$

for  $l \geq 1$ ,  $|m| \leq l$ , where  $\lambda = \sqrt{l(l+1)}$ ,  $j_l$  are spherical Bessel functions and  $Y_{lm}$  spherical harmonics. We

change variables from currents to multipoles in the functional integral and, as the final step in our quantization, integrate over all multipole fluctuations on the two objects weighted by the effective action,

$$\begin{aligned} S_k^{\text{eff}}[\{Q_{lm}^\alpha\}] &= \frac{1}{2} \frac{i}{k} \sum_{lm} \sum_{l'm'} \{Q_{lm}^{1*} U_{lm'l'm'}^- Q_{l'm'}^2 \\ &+ Q_{lm}^{2*} U_{lm'l'm'}^+ Q_{l'm'}^1 + \sum_{\alpha=1,2} Q_{lm}^{\alpha*} [-T^\alpha]_{lm'l'm'}^{-1} Q_{l'm'}^\alpha\}, \end{aligned} \quad (8)$$

with  $Q_{lm}^\alpha = (Q_{M,lm}^\alpha, Q_{E,lm}^\alpha)$ . Let us discuss the terms appearing in Eq. (8) and sketch its derivation.

*Off-diagonal terms* — We need to know the electric fields in Eq. (6) exterior to the source that generates them. They can be represented in terms of the multipoles as  $\mathbf{E}_\beta(\mathbf{x}_\beta) = -k \sum_{lm} Q_{lm}^\beta \Psi_{lm}^{\text{out}}(\mathbf{x}_\beta)$  where  $\Psi_{lm}^{\text{out}}(\mathbf{x}_\beta)$  are *outgoing* vector solutions of the Helmholtz equation in the coordinates of object  $\beta$  [12]. We would like to express the currents  $\mathbf{J}_\alpha^*$  in Eq. (6) also in terms of multipoles. The difficulty in doing so is that the electric field is expressed in terms of *outgoing* partial waves in the coordinates of object  $\beta$ , while according to Eq. (7), the multipoles involve partial waves  $\Psi_{lm}^{\text{reg}}(\mathbf{x}_\alpha)$  that are *regular* at the origin  $\mathcal{O}_\alpha$ , in the coordinates of object  $\alpha$  [12]. Going from the outgoing to the regular vector solutions and changing the coordinate system involves a translation and change of basis which can be expressed as  $\Psi_{lm}^{\text{out}}(\mathbf{x}_\alpha \pm L\hat{\mathbf{z}}) = \sum_{l'm'} U_{lm'l'm'}^\pm \Psi_{l'm'}^{\text{reg}}(\mathbf{x}_\alpha)$  where the *universal* (shape and material independent) matrices  $\mathbb{U}^+$  and  $\mathbb{U}^-$  represent the interaction between the multipoles. For fixed  $(lm)$ ,  $(l'm')$ , they are  $2 \times 2$  matrices (magnetic and electric multipoles), and functions of  $kL$  only. Their explicit form is known but not provided here to save space [13]; they fall off with  $kL$  according to classical expectations for the EM field. Then the electric field becomes  $\frac{1}{ik} \mathbf{E}_\beta(\mathbf{x}_\alpha \pm L\hat{\mathbf{z}}) = \sum_{lm} \phi_{lm}^\beta \Psi_{lm}^{\text{reg}}(\mathbf{x}_\alpha)$  with  $\phi_{lm}^\beta = i \sum_{l'm'} U_{lm'l'm'}^\pm Q_{l'm'}^\beta$ , and the integration in Eq. (6) leads, using Eq. (7), to the off-diagonal terms in Eq. (8).

*Diagonal terms* — The self-action, given by the second term of Eq. (6), is more interesting and more challenging. It can be expressed in terms of multipoles if we use the constraint for the currents, Eqs. (3) and (4). To do so, we first note that in scattering theory one usually knows the incident solution and would like to find the outgoing scattered solution. They are related by the  $S$ -matrix. Here the situation is slightly different. We seek to relate a regular solution  $\mathbf{E}_0(\mathbf{x}_\alpha) = ik \sum_{lm} \phi_{0,lm} \Psi_{lm}^{\text{reg}}(\mathbf{x}_\alpha)$  and the outgoing scattered solution,  $\mathbf{E}_\alpha(\mathbf{x}_\alpha) = -k \sum_{lm} Q_{lm}^\alpha \Psi_{lm}^{\text{out}}(\mathbf{x}_\alpha)$ , generated by the currents in the material — a relation determined by the  $T$ -matrix,  $\mathbb{T}^\alpha \equiv (\mathbb{S}^\alpha - \mathbb{I})/2$  — schematically  $i\mathbf{Q}^\alpha = \mathbb{T}^\alpha \phi_0$  [14, 15]. We face the inverse problem of determining  $\phi_{0,lm}$  for known scattering data  $Q_{lm}^\alpha$ , hence,

$$\phi_{0,lm} = i \sum_{l'm'} [T^\alpha]_{lm'l'm'}^{-1} Q_{l'm'}^\alpha \quad (9)$$

so that the incident field is given in terms of the S-matrix, as indicated in Eq. (4). Next, we express the self-action of the currents inside a body (the second term of Eq. (6)), as  $S_k^\alpha[\mathbf{J}_\alpha] = \frac{1}{2} \int d\mathbf{x}_\alpha [\mathbf{E}\mathbf{D}^* - \mathbf{B}\mathbf{H}^* - (\mathbf{E}_0\mathbf{D}_0^* - \mathbf{B}_0\mathbf{H}_0^*)]$ , the change of the field action that results from placing the body into the *fixed* (regular) incident field  $\mathbf{E}_0 = \mathbf{D}_0$ ,  $\mathbf{H}_0 = \mathbf{B}_0$ , where  $\mathbf{E}$ ,  $\mathbf{H}$  and  $\mathbf{D}$ ,  $\mathbf{B}$  are the new total fields and fluxes in the presence of the body. Using  $\mathbf{D} = \epsilon_\alpha \mathbf{E}$ ,  $\mathbf{H} = \mu_\alpha^{-1} \mathbf{B}$  inside the body and Eq. (3), straightforward manipulations lead to the simple self-action  $S_k^\alpha[\mathbf{J}_\alpha] = -\frac{1}{2ik} \int d\mathbf{x}_\alpha \mathbf{J}_\alpha^* \mathbf{E}_0(\{\mathbf{J}_\alpha, \mathbf{S}^\alpha\})$ . If we substitute the regular wave expansion for  $\mathbf{E}_0$  with coefficients of Eq. (9) and integrate by using Eq. (7), we get Eq. (8).

The T-matrix can be obtained for dielectric objects of arbitrary shape by integrating the standard vector solutions of the Helmholtz equation in dielectric media over the object's surface [15] and both analytical and numerical results are available for many shapes [16]. Hence, for the time being, we shall assume that the elements of the T-matrix are available. The functional integral over multipoles is Gaussian. The resulting partition function is an integral over all frequencies of the determinant of a matrix  $\mathbb{M}$ , with inverse T-matrices along the diagonal and the matrices  $\mathbb{U}^\pm$  off the diagonal. For each  $(lm)$ ,  $\mathbb{M}$  is a  $4 \times 4$  matrix (2 polarizations for 2 objects). The generalization to more than two objects is straightforward. The result is formally infinite but the infinity can be trivially removed by dividing by  $\mathcal{Z}_\infty$ , the partition function with all objects removed to infinite separations, corresponding to setting the off-diagonal terms to zero. Dividing by  $\mathcal{Z}_\infty$  also cancels the functional Jacobian necessary to transform from an integral over sources to an integral over multipoles. After a Wick rotation,  $k \rightarrow i\kappa$ ,

we finally get the Casimir energy

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log \det(\mathbb{I} - \mathbb{U}^- \mathbb{T}^2 \mathbb{U}^+ \mathbb{T}^1) \quad (10)$$

in terms of the matrices introduced in Eq. (8). The dependence of the interaction on distance is completely contained in  $\mathbb{U}^\pm$ , whereas all shape and material dependence comes from the T-matrices. With  $\mathbb{N} \equiv \mathbb{U}^- \mathbb{T}^2 \mathbb{U}^+ \mathbb{T}^1$  it can be written as  $\mathcal{E} = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \text{Tr} \sum_{p=1}^\infty \frac{1}{p} \mathbb{N}^p$  which allows for a simple physical interpretation. The matrix  $\mathbb{N}$  scales with distance  $L$  as  $\sim \exp(-2L\kappa)$  and describes a wave that travels from one object to the other and back, involving one scattering at each object. Hence, we have obtained a multiple-scattering expansion where each elementary two-scattering process, described by  $\mathbb{N}$ , is further decomposed into partial waves. This structure allows for a systematic and exact expansion of the interaction in the inverse distance. At large distance, the interaction is determined by the small  $\kappa$  scaling of the T-matrix,  $T_{lm'l'm'}^\alpha \sim \kappa^{l+l'+1}$ . This shows that  $2p$  scatterings become important at order  $L^{-1-6p}$ , and that partial waves of order  $l$  have to be considered at order  $L^{-5-2l}$ . Hence, in actual computations, the sum over reflections can be cut off at finite  $p$  and the matrix  $\mathbb{N}$  can be truncated to have dimension  $2l(2+l) \times 2l(2+l)$  at partial wave order  $l$  (see below). We note that Eq. (10) applies also to spatially varying but local  $\epsilon_\alpha$  and  $\mu_\alpha$ , since this affects only the T-matrix. Likewise, it can be extended to any other boundary conditions or materials by inserting the appropriate T-matrix.

As a specific example, we consider two identical dielectric spheres. Due to symmetry, the multipoles are decoupled so that the T-matrix is diagonal,

$$T_{lm lm}^{11} = (-1)^l \frac{\pi}{2} \frac{\eta I_{l+\frac{1}{2}}(z) \left[ I_{l+\frac{1}{2}}(nz) + 2nz I'_{l+\frac{1}{2}}(nz) \right] - n I_{l+\frac{1}{2}}(nz) \left[ I_{l+\frac{1}{2}}(z) + 2z I'_{l+\frac{1}{2}}(z) \right]}{\eta K_{l+\frac{1}{2}}(z) \left[ I_{l+\frac{1}{2}}(nz) + 2nz I'_{l+\frac{1}{2}}(nz) \right] - n I_{l+\frac{1}{2}}(nz) \left[ K_{l+\frac{1}{2}}(z) + 2z K'_{l+\frac{1}{2}}(z) \right]}, \quad (11)$$

where the sphere radius is  $R$ ,  $z = \kappa R$ ,  $n = \sqrt{\epsilon(i\kappa)\mu(i\kappa)}$ ,  $\eta = \sqrt{\epsilon(i\kappa)/\mu(i\kappa)}$ , and  $I_{l+\frac{1}{2}}$ ,  $K_{l+\frac{1}{2}}$  are Bessel functions.  $T_{lm lm}^{22}$  is obtained from Eq. (11) by interchanging  $\epsilon$  and  $\mu$ . For all partial waves, the *leading* low frequency contribution is determined by the *static* electric multipole polarizability,  $\alpha_l^E = [(\epsilon - 1)/(\epsilon + (l+1)/l)] R^{2l+1}$ , and the corresponding magnetic polarizability,  $\alpha_l^M = [(\mu - 1)/(\mu + (l+1)/l)] R^{2l+1}$ . Including the next to leading terms, the T-matrix has the structure

$$T_{lm lm}^{11} = \kappa^{2l} \left[ \frac{(-1)^{l-1} (l+1) \alpha_l^M}{l(2l+1)!!(2l-1)!!} \kappa + \gamma_{l3}^M \kappa^3 + \gamma_{l4}^M \kappa^4 + \dots \right],$$

and  $T_{lm lm}^{22}$  is obtained by  $\alpha_l^M \rightarrow \alpha_l^E$ ,  $\gamma_{ln}^M \rightarrow \gamma_{ln}^E$ . The first terms are  $\gamma_{13}^M = -[4 + \mu(\epsilon\mu + \mu - 6)]/[5(\mu + 2)^2] R^5$ ,  $\gamma_{14}^M = (4/9)[(\mu - 1)/(\mu + 2)]^2 R^6$ , and  $\gamma_{13}^E$ ,  $\gamma_{14}^E$  are obtained again by the replacement,  $\mu \rightarrow \epsilon$ . Now we can apply our general formula in Eq. (10) to two dielectric spheres with

center-to-center distance  $L$ . For simplicity, we restrict to two partial waves ( $l = 2$ ) and two scatterings ( $p = 1$ ), which yields the exact Casimir energy to order  $L^{-10}$ . Matrix operations are performed with **Mathematica**, and we find the interaction

$$\mathcal{E} = -\frac{\hbar c}{\pi} \left\{ \left[ \frac{23}{4} ((\alpha_1^E)^2 + (\alpha_1^M)^2) - \frac{7}{2} \alpha_1^E \alpha_1^M \right] \frac{1}{L^7} + \frac{9}{16} [\alpha_1^E (59\alpha_2^E - 11\alpha_2^M + 86\gamma_{13}^E - 54\gamma_{13}^M) + E \leftrightarrow M] \frac{1}{L^9} + \frac{315}{16} [\alpha_1^E (7\gamma_{14}^E - 5\gamma_{14}^M) + E \leftrightarrow M] \frac{1}{L^{10}} + \dots \right\}, \quad (12)$$

where  $E \leftrightarrow M$  indicates terms with exchanged superscripts. The leading term,  $\sim L^{-7}$ , has precisely the form of the Casimir-Polder force between two atoms [4], in-

cluding magnetic effects [5]. The higher order terms are new, and provide the first systematic result for dielectrics with strong curvature. There is no  $\sim 1/L^8$  term.

The limit of perfect metals follows for  $\epsilon \rightarrow \infty$ ,  $\mu \rightarrow 0$ . Then higher orders are easily included, yielding an asymptotic series

$$\mathcal{E} = -\frac{\hbar c}{\pi} \frac{R^6}{L^7} \sum_{n=0}^{\infty} c_n \left(\frac{R}{L}\right)^n, \quad (13)$$

where the first 10 coefficients are  $c_0 = 143/16$ ,  $c_1 = 0$ ,  $c_2 = 7947/160$ ,  $c_3 = 2065/32$ ,  $c_4 = 27705347/100800$ ,  $c_5 = -55251/64$ ,  $c_6 = 1373212550401/144506880$ ,  $c_7 = -7583389/320$ ,  $c_8 = -2516749144274023/44508119040$ ,  $c_9 = 274953589659739/275251200$ . This series is obtained by expanding in powers of  $R/L$  and frequency  $\kappa$ , and does not converge for any fixed  $R/L$ . To obtain the energy at all separations, one has to compute Eq. (10) without these expansions. This is done by truncating the matrix  $\mathbf{N}$  at a finite multipole order  $l$ , and computing the determinant and the integral numerically. The result is shown in Fig. 1 for perfect metal spheres. Our data indicate that the energy converges as  $e^{-\delta(L/R-2)^l}$  to its exact value at  $l \rightarrow \infty$ , with  $\delta \sim \mathcal{O}(1)$ . Our result spans all separations between the Casimir-Polder limit for  $L \gg R$ , and the proximity force approximation (PFA) for  $R/L \rightarrow 1/2$ . At a surface-to-surface distance  $d = 4R/3$  ( $R/L = 0.3$ ), PFA overestimates the energy by a factor of 10. Including up to  $l = 32$  and extrapolating based on the exponential fit, we can accurately determine the Casimir energy down to  $R/L = 0.49$ , *i.e.*  $d = 0.04R$ . A similar numerical evaluation can be also applied to dielectrics [10].

We have developed a systematic method for computing the EM Casimir interaction between compact dielectric objects of arbitrary shapes. Casimir interactions are completely characterized by the S-matrices of the individual bodies. We have computed the force between spheres for arbitrary separations, generalizing previous results that applied only in singular limits. Our method allows for the first time a description of the Casimir interaction from atomic-scale particles (Casimir-Polder limit) up to macroscopic objects at short separations (PFA limit). For more complicated shapes and multiple objects, it would be interesting to probe the dependence on the relative orientations of non-spherical objects and corrections to pair-wise additivity. Our approach can be applied at finite temperatures and extended to the computation of correlation functions, energy densities, and the density of states and may prove also useful to obtain thermal (classical) fluctuation forces.

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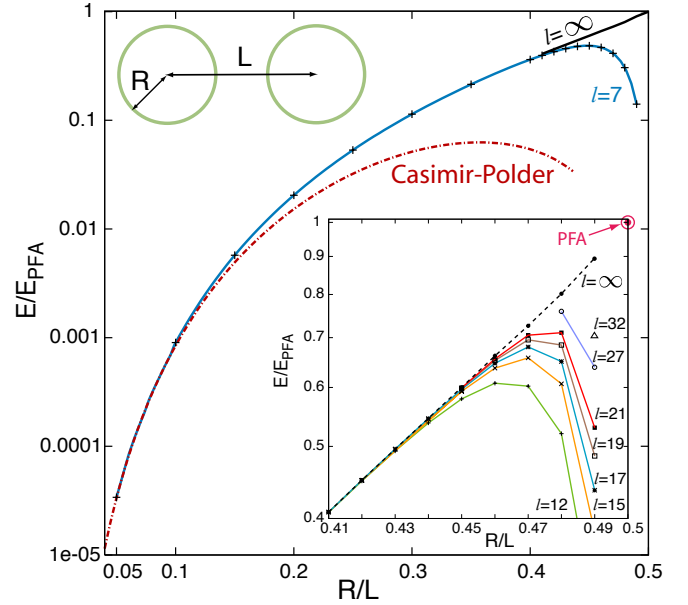


FIG. 1: Casimir energy of two metal spheres, divided by the PFA estimate  $\mathcal{E}_{\text{PFA}} = -(\pi^3/1440)\hbar c R/(L-2R)^2$ , which holds only in the limit  $R/L \rightarrow 1/2$ . The label  $l$  denotes the multipole order of truncation. The curves  $l = \infty$  are obtained by extrapolation. The Casimir-Polder curve is the leading term of Eq. (13). Inset: Convergence at short separations.

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